MATH543 : METHODS OF APPLIED MATHEMATICS

LECTURE NOTES

FALL 2012

MATH 543 APPLIED MATHEMATICS I

(2000-2011)

2012 Fall Semester

Math543: Applied Mathematics I

Text Books :

1. P. Dennery and A. Krzywicki, "Mathematics for Physicists", Harper and Row, 1967.
2. F. B. Hildebrand, " Methods of Applied Mathematics", second edition, Prentice Hall.
3. Sadri Hassan, "Mathematical Physics: A Modern Introduction to its Foundations", Springer Verlag, New York, 1999.
4. J. David Logan, "Applied Mathematics", John Willey and Sons, Inc., New York, 1997 (Second Edition).
5. Haaser and Sullivan , "Real Analaysis", The University Series in Undergraduate Mathematics.
6. W. E. Boyce and R. C. DiPrima, " Elementary Differential Equations and Boundary Value Problems",
Sixth Edition, John Wiley and Sons, Inc.

some subjects in applied mathematics

Course Schedule

Tuesday 13.40-15.30 (SA141)
Thursday 15.40-17.30 (SA141)

Exams

For previous exams see prev.exams

(30%) First Midterm Exam 2007: pdf file , 2008 , 2010 , 2011 **OCTOBER 30**
(30%) Second Midterm Exam 2006: pdf file , 2008 , 2011 **DECEMBER 11**
(40%) Final Exam 2007: pdf file , 2008 , 2010 , 2011 **January xx, 2013**

For exam results please see math543exams

In the exams you will be responsable from DK, Logan, Lecture notes and the assigned exercises below

Students

Contents of Applied Mathematics I

For preparation read the last Chapter of Haaser and Sullivann "Real Analysis"

Assigned Exercises

1. set 1 pdf file , ps file

2. set 2 pdf file , ps file
3. set 3 pdf file , ps file
4. set 4 pdf file , ps file
1st exam pdf file
5. set 5 dvi file , ps file
6. set 6 dvi file , ps file
7. set 7 dvi file , ps file
2nd exam with solutions pdf file
8. set 8 dvi file , ps file
9. set 9 pdf file
10. set 10 pdf file
Final exam (pdf file)

Subjects covered

1. September 26
- Funcion Space, completeness, square integrable functions.

Chapter III of Dennery, Chapter 5 (Hilbert Spaces) of Sadri Hassan
assigned exercises, set 1 pdf file ,
Homework Set 1 pdf file ,
2. October 3
- Orthogonal set of vectors and the Bessel inequality.
- Basis and the Parseval's relation
- Weierstrass theorem

assigned exercises, set 2, pdf file ,
3. October 10
- Classification of orthogonal polynomials
- Classical orthogonal polynomials

Homework Set 2 pdf file ,
assigned exercises, set 3 pdf file
Spherical Harmonics in D Dimensions
4. October 17
- Trigonometric series
- Generalized functions

assigned exercises, set 4 pdf file
5. October 24
- Fourier transform of Generalized Functions
- Second order differential equations: Fundamental solutions, method of variation of constants.

6. October 31
- Generalized Greens identiy: Adjoint operators and adjoint bounday conditions.

(2007)
7. November 14
- The Method of Green's Function
- The Method of Green's Function (applications)

First Midterm Exam pdf file (2006) ,
Homework Set 3 pdf file , assignaed exercises, set 5 pdf file
8. November 21
- The Sturm Liouville Problem
- Classification of singular points

9. November 28
- The Frobenius method: The series solutions of linear DEs

assigned exercises, set 6 pdf file ,
(**Assigned exercises** : Solve also the problems of the Sections 5.4-5.8 of Boyce and DiPrima)
10. December 5

- The Frobenius method: The series solutions of linear DEs

(**Assigned exercises** : Solve also the problems of the Sections 5.4-5.8 of Boyce and DiPrima)

Second Midterm Exam pdf file

11. December 12

- Fuchsian Differential Equations
- The Hypergeometric Function

assigned exercises, set 7 pdf file

- Solutions of DEs by Integral Representations
- Integral Representations of Hypergeometric Functions

assigned exercises, set 8 pdf file

Homework Set IV: From set 7 Problems 12,13 and from set 8 Problems 2,3

(Due December 28)

12. December 19

- Regular Perturbations
- Poincare-Lindstedt Method

Assigned exercises, set 9 pdf file

13. December 26

- Poincare-Lindstedt Method
- Singular Perturbations
- Boundary Layer Problems

Second Midterm Exam pdf file

14. January 2

- WKB Approximation
- Asymtotic Expansion of Integrals
- Calculus of Variations

15. December xx

- Calculus of Variations
- Necessary Condition
- Euler-Lagrange Equations
- Lagrange functions depending on higher derivatives
- Null Lagrange functions
- Lagrange function of given DE
- Lagrange function with several dependent variables

Assigned exercises, set 10 pdf file

end of the semester

16. December 28

- Lagrangeans with Several Functions
- Isoperimetric Problems
- Lagrange Function of a given DE

Subjects to be covered

Ch.1. Function Space, Orthogonal Polynomials and Fourier Analysis (Dennery and Krzywicki)

- Space of Continuous Functions
- Expansion in Orthoganal Functions
- The Classical Orthogonal Polynomials
- Trigonometric Series
- Generalized Functions
- Linear Operators in Infinite Dimensional Space

First Midterm

Ch.2. Ordinary Differential Equations (Dennery and Krzywicki)

- Second Order Differential Equations
- Generalized Green's Identity

- Green's Functions
- The Sturm-Liouville Problem
- Series Solution of Linear Differential Equations
- The Hypergeometric Function

Second Midterm

Ch.3. Perturbation Methods (Logan)

- Regular Peturbations
- Singular Perturbation
- Boundary Layer Analysis
- Applications
- The WKB Approximation
- Asymptotic Expansions of Integrals

Final Exam

Ch.4. Calculus of Variations (Logan and Hildebrand)

- Variational Problems
- Necessary Conditions for Extrema
- The simplest problem
- Generalizations
- Isoperimetric Problems

Course Syllabus of Math543

01. Sept. 24 - Function Space
02. Oct. 01 - Function Space
03. Oct. 08 - Function Space
04. Oct. 15 - Ordinary Differential Equations
05. Oct. 22 - Ordinary Differential Equations
06. Oct. 29 - Ordinar Differential Equations
07. Nov. 05 - Ordinary Differential Equations
08. Nov. 12 - Perturbation Methods
09. Nov. 19 - Perturbation Methods
10. Nov. 26 - Perturbation Methods
11. Dec. 03 - Perturbation Methods
12. Dec. 10 - Calculus of Variations
13. Dec. 17 - Calculus of Variations
14. Dec. 21 - Calculus of Variations

Last update November 2003

End of Applied1's Home Page

MATH543: Methods of Applied Mathematics

Fall 2012 Lecture Notes

Main References are

1. P. Dennery and A. Krzywicki, "Mathematics for Physicists", Harper and Row, 1967

   In Lectures 1-4 we mainly follow this reference

2. F. B. Hildebrandt, "Methods of Applied Mathematics" second edition, Prentice Hall

3. J. David Logan, "Applied Mathematics" John Wiley and Sons, Inc., New York, 1997

Metin Gürses

Lecture 1 (Function Spaces).

1. Linear Spaces (Linear vector spaces)
   a) finite dimensional
   b) Infinite dimensional cases

2. Metric Spaces
   a) finite dimensional
   b) Infinite dimensional cases.

3. The normed spaces.

~~Inn~~

4. Inner product Spaces.
   a) finite dimensional
   b) Infinite dimensional cases.

1. <u>Linear Spaces:</u> A linear vector space X over a field F is a set of elements togather with a function, called addition, from $X \times X$ to X and a function, scalar multiplication, from $F \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in F$

1. $(x+y)+z = x+(y+z)$ "associativity"
2. $x+y = y+x$ "commutativity"
3. There is an element 0 in X such that
   $x + 0 = x$ for all $x \in X$.
4. For each $x \in X$ there exists an element $-x \in X$ such that $x+(-x) = 0$
5. $\alpha(x+y) = \alpha x + \alpha y$
6. $(\alpha+\beta)x = \alpha x + \beta x$
7. $\alpha(\beta x) = (\alpha\beta)x$
8. $1 \cdot x = x$.

1-6 imply that X is an abelian group under addition.

5-8 relate the operation of scalar multiplication to addition in X and to addition and multiplication in F.

a) Finite Dimensional case: $V_n(\mathbb{R}) = X$

$x = (x^1, x^2, \ldots, x^n)$, $y = (y^1, y^2, \ldots, y^n)$

$z = (z^1, z^2, \ldots, z^n)$

let i) $x + y = (x^1, x^2, \ldots, x^n) + (y^1, y^2, \ldots, y^n)$
$= (x^1 + y^1, x^2 + y^2, \ldots, x^n + y^n)$

ii) $\alpha X = \alpha(x^1, x^2, \ldots, x^n)$
$= (\alpha x^1, \alpha x^2, \ldots, \alpha x^n)$

Then we can prove that $V_n(\mathbb{R})$ is a linear vector space over $\mathbb{R}$. One can also prove that $V_n(\mathbb{C})$ is a linear vector space ~~so~~ over complex numbers.

b) Infinite dimensional case : Function space. The set of all functions from a nonempty set into a field F with addition and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x).$$

$$(\alpha f)(x) = \alpha f(x).$$

where $x \in [a,b]$. One can show that the space of continuous functions form a linear space. A sketch of the proof is as follows.

a) Adding two continuous functions one obtains a continuous function

b) multiplication by a number of a continuous function gives again a continuous function.

c) The function that is zero identically in $[a,b]$, when added to a continuous function doesn't alter this function

d) for any continuous function $f(x)$

there exist a function $-f(x) = (-1)f(x)$ which satisfies $f(x) + [(-1)f(x)] = 0$.

$\Rightarrow$ continuous functions over $[a,b]$ form a linear space.

<u>Notation</u>: $f(x)$ or $|f\rangle$ for the same function (This is notation of DK).

2. Metric Spaces. A set S is called a <u>metric space</u> if a real and positive number $\rho(a,b)$ is associated with any pair of its elements $a, b \in S$ and if

(a) $\rho(a,b) = \rho(b,a)$ "symmetry"

(b) $\rho(a,b) \geq 0$, $\rho(a,b) = 0$ only when $a = b$.

(c) $\rho(a,b) + \rho(b,c) \geq \rho(a,c)$ "triangular inequality"

Then the number $\rho(a,b)$ is called the <u>distance</u> between a and b.

Conditions (a) and (b) mean that the distance from a to b is the same that from b to a, and that the distance vanishes only when two elements coincide. The last condition is known as the triangle inequality.

Examples: i) Any set of points on a plane is a metric space if $\rho(a,b)$ is the standard distance between the points a and b.
ii) This notion of a distance between the elements of a set is now extended to the case where the set constitute a linear vector space. Let $|a\rangle$, $|b\rangle$ be elements of a linear space let $\langle a|b\rangle$ define a <u>scalar product</u> (inner product) then the length (norm) of $|a\rangle$ is $(\langle a|a\rangle)^{1/2}$ and the distance from $|a\rangle$ to $|b\rangle$ is

$$\rho(a,b) = (\langle a-b|a-b\rangle)^{1/2}.$$

Let $X = V_n(\mathbb{R})$ and let $|e_i\rangle$ be a basis of $V_n(\mathbb{R})$, $|a\rangle = \sum_i a_i |e_i\rangle \Rightarrow$

$$\rho(a,b) = \left(\sum |a_i - b_i|^2\right)^{1/2}.$$

3. Normed Spaces: The vector space X is called a normed space if there is a function (called norm) denoted $\|\cdot\|: X \to \mathbb{R}^+$ which for all $x \in X$ it satisfies the following properties.

1. $\|x\| \geq 0$ and $\|x\| = 0$ implies $x = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$ "triangular inequality"

If there is a norm in a vector space we automatically have a metric (i.e a norm linear vector space is also a metric space). The converse is not true.

Examples:

[l^2] (i) Let $\{x_n\}$ be a set of all sequences. This set with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ is a normed space which is denoted as l^2 with.

$$||x|| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}.$$

One can show that all three properties (1-3) are satisfied.

[l^p] (ii) Let

$$||x|| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

and assuming $\sum_{n=1}^{\infty} |x_n|^p < \infty$, this is a normed space denoted as l^p.

[L^p] (iii). Let us consider the space of functions on the interval $[a,b]$. $L^p[a,b]$ is a normed space with

$$||f(x)|| = \left(\int_0^b |f^p(x)| dx \right)^{1/p}$$

Consider the function $f(x) = x^{-1/3}$ over $(0,1]$. which belong to $L^1[0,1]$ and $L^2[0,1]$ but not $C[0,1]$ nor to $L^p[0,1]$ if $p \geq 3$

proof:

Hence if $f(x) = x^{-1/3}$, $x \in (0,1]$

$\Rightarrow f(x) \in C(0,1]$ which has no maximum value

and

$$\|f\|_p = \left(\int_0^1 |x|^{-p/3} dx \right)^{1/p} = \left(\frac{1}{1-p/3} x^{1-p/3} \Big|_0^1 \right)^{1/p}$$

$$= \frac{1}{1-p/3} \left[\left(1 - \lim_{x \to 0} x^{1-p/3} \right) \right]^{1/p}$$

$$\lim_{x \to 0} x^{1-p/3} = \begin{cases} 0 & p = 1, 2. \\ 1 & p = 3. \\ \text{not defined for} & p > 3. \end{cases}$$

$\|f_p\|_p$ defined only for $p = 1$ and $p = 2$.

$\Rightarrow f \in L^1(0,1]$ and $f \in L^2(0,1]$.

(iv) $C[a,b]$ is vector space with norm

$$\|f\| = \max_{x \in [a,b]} |f(x)|$$ "supremum" or "uniform norm"

The above examples does not belong to this space in $[0,1]$.

4. Inner Product Spaces : An inner product $\langle\ ,\ \rangle$ is a bilinear operation on $X \times X$ into the real (or complex) numbers with properties , $x, y, z \in X$, $\alpha \in \mathbb{R}$.

1. $\langle x, y\rangle = \langle y, x\rangle$.
2. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$
3. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0$ iff $x=0$

Schwarz inequality holds.

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$

A linear space with an inner product $\langle\ ,\ \rangle$ is called an inner product space

- A complete inner product space is called a Hilbert space , like ℓ^2 and $L^2[a,b]$.
- A complete normed space is called a Banach space like $L^p[a,b]$, $p \neq 2$.

Some definitions

1. Definition: A sequence $\{x_n\}$ in a set S (a linear space) is said to have a limit $x \in S$ or to converge to x $(\lim_{n\to\infty} x_n = x)$ if for every $\varepsilon > 0$ there exist an integer N (depending on ε) so that for any $n \geq N$

$$\rho(x_n, x) < \varepsilon$$

2. Definition: A sequence $\{x_n\}$ in a set S is called a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N (depending on ε) such that for every $m, n > N$, $\rho(x_m, x_n) < \varepsilon$

3. Definition: A normed linear space S is "complete" if every Cauchy sequence in S is convergent in S

Examples:

a) real numbers are complete

b) rational numbers are not complete
Let $x_n = (1+\frac{1}{n})^n$. $S = 2, \frac{9}{4}, \frac{64}{27}, \ldots$
$\lim_{n\to\infty} x_n = e$ not in the rational numbers.

~~Hence not all normed linear spaces are complete~~

The space of continuous functions is complete wrt the uniform norm but not wrt to the L^2 norm.

- Infinite dimensional case: Let $f, g \in \mathcal{F}$ where $\mathcal{F}$ be set of continuous function over an interval $[a,b]$, $\mathcal{F} = C[a,b]$. We can define an inner product

$$\langle f | g \rangle = \int_a^b \bar{f}(x)\, g(x)\, w(x)\, dx.$$

where $w(x)$ is the weight function with $w(x) > 0 \quad \forall x \in [a,b)$ then.

$$\rho(|f\rangle, |g\rangle) = \left(\int_a^b |f(x) - g(x)|^2 w(x)\, dx \right)^{1/2}$$

Cauchy-Schwarz inequality $|\langle f|g\rangle|^2 \leq \langle f|f\rangle \langle g|g\rangle$ in terms of this product is.

$$\left| \int_a^b \bar{f}(x)\, g(x)\, w(x)\, dx \right|^2 \leq \left(\int_a^b w(x)\, |f(x)|^2 dx \right) \left(\int_a^b w(x)\, |g(x)|^2 dx \right)$$

<u>Lemma 1</u>. Consider an infinite sequence of elements of a metric space: $a_{(1)}, a_{(2)}, \ldots, a_{(k)}, \ldots$ suppose that there exists an element a of the space such that the distance $\rho(a, a_{(k)})$ $(k = 1, 2, \ldots, n)$ between the members of the sequence become smaller and smaller as k increases and in the limit as $k \to \infty$ tends to zero

$$\lim_{k\to\infty} \rho(a, a_{(k)}) = 0$$

then a is unique.

<u>proof</u>: use the ~~[illegible]~~ or triangular inequality say that there exists another element b such that

$$\lim_{k\to\infty} \rho(b, a_{(k)}) = 0$$

Then using the triangular inequality

$$\rho(a, a_{(k)}) + \rho(b, a_{(k)}) \geq \rho(a,b) \qquad \forall k$$

let $k \to 0 \Rightarrow \rho(a,b) = 0 \quad \Rightarrow b = a.$

- Finite Dimensional case: An N-dimensional vector space with distance between two vectors $|a\rangle$ and $|b\rangle$ given as.

$$\rho(a,b) = \left(\langle a-b|a-b\rangle\right)^{1/2}$$

$$= \left(\sum_{i=1}^{N} |a_i - b_i|^2\right)^{1/2}$$

Note that, in N-dimensional vector space there exist a basis $|e_i\rangle$ such that

$$|a\rangle = \sum_{i=1}^{n} a_i |e_i\rangle$$

with $\langle e_i | e_j \rangle = \delta_{ij}$ $\forall\, i,j = 1,2,\ldots,N$

for any vector $|a\rangle \in V_N(\mathbb{C})$.

Finite dimensional vector spaces have an important property which, as we shall see later, is not shared by all linear vector spaces, and in particular by the space of continuous functions.

Let $|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle, \ldots \in S_N$. (N-dimensional vectors space S_N). Satisfy the condition

$$\lim_{k,l \to \infty} \rho(|a_k\rangle, |a_l\rangle) = 0$$

(called Cauchy sequence) it is not difficult to prove that this implies that there exist a vector $|a\rangle \in S_N$ to which

the sequence $|a_1\rangle, |a_2\rangle, \dots, |a_n\rangle, \dots$ converges to
i.e

$$\lim_{k\to\infty} \rho(|a_k\rangle, |a\rangle) = 0$$

($|a\rangle$ is unique we know).

<u>proof</u>: Cauch Criteria implies that

$$\sum_{i=1}^{N} |a^i_{(k)} - a^i_{(l)}|^2 < \varepsilon$$

Use the definition of the limit.

$$\lim_{\substack{k,l\\ \to\infty}} \sum_{i=1}^{N} |a^i_{(k)} - a^i_{(l)}|^2 = 0.$$

This means that, given an arbitrary positive real number $\varepsilon > 0$ there exist a number L such that

$$\sum_{i=1}^{N} |a^i_{(k)} - a^i_{(l)}|^2 < \varepsilon \qquad \forall\, k,l > L$$

or $$|a^i_{(k)} - a^i_{(l)}|^2 < \varepsilon \qquad \forall\, k,l > L.$$

see page (16) for more detail

From Cauchy Criterion this means that there exist a real number a^i such that the infinite sequence $a^i_{(1)}, a^i_{(2)}, \dots, a^i_{(n)}, \dots$ converges to (property of the real numbers). On the other hand this implies the existence of an element $|a\rangle = \sum_{i=1}^{N} a_i |e_i\rangle$. (see page 16)

This property, known as the completeness, is not true for an arbitrary linear vector space. That is, the Cauchy condition

$$\lim_{k,l} \rho(|a_k\rangle, |a_l\rangle) = 0$$

does not necessarily imply, in general, that there exists a vector $|a\rangle$ such that

$$\lim_{k\to\infty} \rho(|a\rangle, |a_k\rangle) = 0$$

Spaces having this property is called "complete" or complete vector spaces

A simple example is the space of continuous functions on some interval. Later we shall give a simple example showing that the space of continuous functions is not complete.

Real numbers $\mathbb{R}$. A sequence $\{a_1, a_2, \dots, a_k, \dots\}$ has Cauchy property

$$\lim_{k,l} |a_k - a_l| = 0$$

or $$|a_k - a_l| < \varepsilon \quad \forall\, k, l > M$$

$\Rightarrow$ there exists an element a real number ~~of the sequence~~

$$\lim_{k\to\infty} |a_k - a| = 0$$

$\Rightarrow$ for finite dimensional vector space:

$$\rho(|a_k\rangle, |a_l\rangle) = (\langle a_k| - \langle a_l|)(|a_k\rangle - |a_l\rangle)$$

$$|a_k\rangle = \sum_i a_k^i |e_i\rangle$$

$$\rho(|a_k\rangle, |a_l\rangle) = \sum_{i=1}^{N}{}' |a_k^i - a_l^i|^2$$

$$\lim_{k,l} \sum^{N}{}' |a_k^i - a_l^i|^2 = 0$$

$$\Rightarrow \sum{}' |a_k^i - a_l|^2 < \varepsilon \quad \forall\, k, l > M$$

or $$|a_k^i - a_l^i|^2 < \varepsilon \quad \forall\, k, l > M \quad \forall i$$

$$\Rightarrow \exists\, a^i \text{ such that } \lim_{k\to\infty} |a_k^i - a^i|^2 = 0$$

$$\Rightarrow \Rightarrow \lim_{k=\infty} \sum |a_k^i - a^i|^2 = 0 \Rightarrow \lim_{k\to\infty} \rho(|a_k\rangle, |a\rangle) = 0$$

Some examples

<u>Example</u>: let $X = \mathbb{R}$, and define $\|x\| = |x|$, the familiar absolute value function

<u>Example</u> let $X = \mathbb{C}$, where the scalar field is also $\mathbb{C}$, use

$\|x\| = |x|$, where

$|x|$ has its usual meaning for a complex number x. If $x = a + ib$

$\Rightarrow \|x\| = \sqrt{a^2+b^2}$

<u>Example</u>: Let $X = \mathbb{C}$ and take the scalar field to be $\mathbb{R}$. The terminology we adopt is that X is a <u>real vector space</u>, since the scalar field is <u>real</u>

<u>Example</u> let $X = \mathbb{R}^n$. Here the elements of X are n-tuples of real numbers that we can display in the form

$$x = \{x(1), x(2), \dots, x(n)\}$$

A usual norm is defined by the equation

$$\|x\|_\infty = \max_{1 \le i \le n} |x(i)|$$

(sup norm).

Example let $X = \mathbb{R}^n$ and denote

$$\|x\| = \sum_{i=1}^{n} |x(i)| \equiv \|x\|_1$$

a theorem: $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$\|x\|_\infty = |x(k)|. \quad \left(= \max_{i \le n} |x(i)| \right)$

where $|x(k)| \geq |x(i)| \quad \forall i = 1, 2, \ldots, n$

$$\|x\|_2 \geq |x(k)| = \|x\|_\infty$$

$$\|x\|_2 \le \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

since $x(i)^2 \le x(k)^2$

$$\|x\|_1 \le \sqrt{n}\, \|x(k)\|_\infty$$

$\Rightarrow$

$$\|x\|_\infty \le \|x\|_1 \le \sqrt{n}\, \|x\|_\infty$$

or

$$\boxed{\frac{1}{\sqrt{n}} \|x\|_1 \le \|x\|_\infty \le \|x\|_2}$$ Theorem

Example ① Let X be the set of all real-valued continuous functions defined on a fixed compact interval $[a,b]$. The norm usually employed here is

$$\|x\|_\infty = \max_{a \le s \le b} |x(s)|$$

(The notation $\max_{a \le s \le b} |x(s)|$ denotes the maximum of the expression and s runs over the interval $[a,b]$.). The space X described here is often denoted by $C[a,b]$. This norm is called "sup" norm.

$$\sup |x(s)| \equiv \max_{a \le s \le b} |x(s)|$$

<u>Example</u>: Let $X = \ell$, the space of all sequences in $\mathbb{R}$

$$x = \{x(1), x(2), \ldots, x(n), \ldots\}$$

in which only a <u>finite number of terms are nonzero</u>. (The number of nonzero terms is not fixed but can vary with different sequences)

Define $\|x\| = \max_n |x(n)|$

<u>Example</u>: Let $X = \ell_\infty$, the space of all real sequences x for which $\sup_n |x(n)| < \infty$. Define $\|x\|$ to be that supremum.

$$\|x\| = \sup_n |x(n)| \equiv \max_n |x(n)|$$

<u>Example</u>: Let $X = P$ the space of all polynomials having real coefficients. A typical element of P is a function x having the form

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

~~other~~ possible norms on P are

$$x \to \max_i |a_i|$$

$$x \to \max_{0 \le t \le 1} |x(t)|.$$

$$x \to \int_0^1 |x(t)|\, dt$$

$$x \to \left(\sum_{k=0}^{n} |x(k)|^3 \right)^{1/3}$$

<u>Example:</u> Let $X = \mathbb{R}^n$, and use the familier <u>Euclidean norm</u>, defined by

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x(i)|^2 \right)^{1/2}$$

where $x = (x_1, x_2, \dots, x_n)$.

Definition: A sequence $\{x_n\}$ in a normed linear space X is said to have the *Cauchy Property* or to be a *Cauchy sequence* if

$$\lim_{n} \sup_{\substack{i \geq n \\ j \geq n}} \|x_i - x_j\| = 0$$

If every Cauch sequence in the space X is convergent, then the space X is said to be *complete*

Definition. A complete normed linear space is called a Banach Space

Remark: completeness is important in constructing solutions to a problem by taking the limit of successive ~~a appr~~ approximations

<u>Theorem</u>: The space $C[a,b]$ with norm $\|x\| = \max_s |x(s)|$ is a Banach space

<u>proof</u>: Let $\{x_n\}$ be a Cauch sequence in $C[a,b]$. Then for each s, $[x_n(s)]$ is a a Cauchy sequence in R. Since R is complete, this later sequence converges to a real number that we denote by $x(s)$. The function x thus defined must be continuous and we must also show that

$$\|x_n - x\| \to 0 .$$

Let t be fixed as the point which the continuity is to be proven. We mean that let x be continuous a point $t \in [a,b]$. Then

$$|x(s) - x(t)| \le |x(s) - x_n(s)| + |x_n(s) - x_n(t)| + |x_n(t) - x(t)|$$

let $\varepsilon > 0$ and select N so that

$\|x_n - x_m\| \leq \frac{\varepsilon}{3}$ for all $n, m \geq N$

(Cauchy property) x_n being the Cauchy seq.

hence for all $m, n \geq N$

$\| x_n(s) - x_m(s) \| \leq \varepsilon/3$

by letting $m \to \infty$ we get

$\| x_n(s) - x(s) \| < \varepsilon/3 \quad \forall s \in [a,b]$

This shows that $\|x_n - x\| \leq \varepsilon/3$
and the sequence $\|x_n - x\|$ converges to zero

by the continuity of x_n there exists
a $\delta > 0$ such that

$| x_n(s) - x_n(t) | \leq \varepsilon/3$ whenever $|t-s| < \delta$

$$\Rightarrow \quad |x(s) - x(t)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

when $|s-t| < \delta$

hence $x \in C[a,b]$

$\Rightarrow$ $C[a,b]$ is complete $\Rightarrow$ Banach.

Math 543. Lecture 2

1. Definition of completeness an example
2. $L^2_w(a,b)$ "The square integrable functions"
3. The Riesz-Fisher Theorem.
4. Expansion in orthogonal polynomials.

   a) Polynomial functions
   b) The general case: The Bessel inequality
   c) Definition of basis
   d) A theorem on Fourier coefficients.
   e) Parseval's relation
   f) The generalization of the notion of a basis.

1. Definition of completeness and an example

<u>Definition</u>: Let V be a linear vector space. Let $|\alpha_k\rangle$, $k = 1, 2, \ldots$ be an infinite sequence. If the Cauchy convergence condition

$$\lim_{k,l \to \infty} \rho(|\alpha_k\rangle, |\alpha_l\rangle) = 0 \qquad (2.1)$$

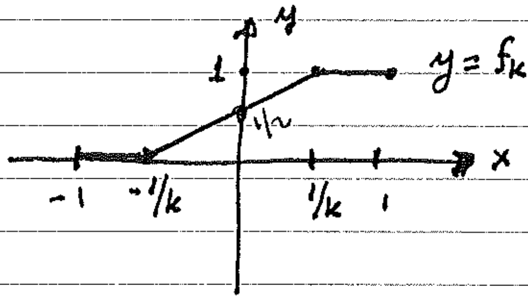
implies also an existence of a vector $|\alpha\rangle$ <u>in V</u> such that the sequence convergence to, i.e

$$\lim_{k \to \infty} \rho(|\alpha\rangle, |\alpha_k\rangle) = 0 \qquad (2.2)$$

Then V is called a "complete" vector space

<u>Remark</u>: For finite dimensional vector spaces (2.1) implies (2.2). Hence they are complete. The infinite dimensional vector spaces may not be a complete vector space. An example is F: the space of ~~function of~~ continuous functions on an interval $[a, b]$.

<u>Example</u>: Consider the following set of functions f_k with $w = 1$ and $I = [a, b]$.
$[-1, 1]$



, $k = 1, 2, \ldots$

$\{f_1, f_2, \ldots, f_k, \ldots\}$
a sequence of continuous functions over $[-1, 1]$ with $w = 1$.

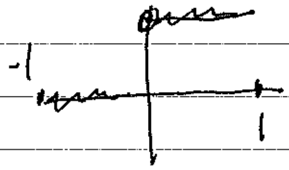
$$f_k = \begin{cases} 0 & -1 \leq x < -\frac{1}{k} \\ \frac{kx+1}{2} & -\frac{1}{k} < x < \frac{1}{k} \\ 1 & \frac{1}{k} < x \leq 1 \end{cases}, \quad k = 1, 2, \ldots$$

i) You can prove that

$$\lim_{k,l \to \infty} \rho^2(|f_k\rangle, |f_l\rangle) = \int_{-1}^{1} |f_k(x) - f_l(x)|^2 dx \xrightarrow[k,l \to \infty]{} 0$$

ii) You can show that this sequence converges to a function $f(x)$

$$f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & -1 \leq x < 0 \end{cases}$$



where

$$f(x) = \lim_{k \to \infty} f_k(x)$$

which is not a continuous function, hence $f \notin F$. This means that there doesnot exist a function $g(x)$ satisfying (in F)

$$\lim_{k \to \infty} \rho(|g\rangle, |f_k\rangle) = 0$$

iii) For the proof: take the intervals $[-1, 0)$ and $(0, 1]$ sepately in one of the

$g(x) = 0$ and in other $g(x) = 1$

due to the uniqueness in each interval $g(x) \notin F$

3. The Riesz-Fisher Theorem:

- We saw that the space of continuous functions is not a complete vector space. Later we shall see the importance of the "completeness".

- Hence to make F complete we change the properties of the function space. Instead of the continuous functions over $[a,b]$ consider square integrable functions, i.e

$$\|f\|^2 \equiv \int_a^b |f(x)|^2 w(x)\, dx < \infty$$
$$= \langle f, f \rangle$$

Such a space is denoted as $L^2_w(a,b)$. Any vector $|g\rangle$ in $L^2_w(a,b)$ has finite norm

$$\|g\|^2 = \langle g|g\rangle = \int_a^b w(x)\, |g(x)|^2 dx < \infty.$$

- $L^2_w(a,b)$ contains F, not only continuous functions, piece-wise continuous functions (jump discontinuities) are also in $L^2_w(a,b)$

L denotes Lebesque

- The integrals we consider here all Riemann integrals although L comes from Lebegue

Some properties:

i) $f, g \in L^2_w(a,b) \Rightarrow f+g \in L^2_w(a,b)$

<u>proof</u>: first remember the Cauchy-Schwarz inequality

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle \langle g|g\rangle$$

or

$$\left| \int_a^b \bar{f}(x)\, g(x)\, w(x)\, dx \right|^2 \leq \left| \int_a^b w\, |f|^2 dx \right| \left| \int_a^b w |g|^2 dx \right|$$

$$\leq \|f\|^2 \|g\|^2 < \infty$$

$\Rightarrow$

$$|\langle f | g \rangle| < \infty$$

Secondly consider $\left(\langle f|g\rangle \equiv \int_a^b w(x)\, \bar{f}(x)\, g(x)\, dx \right)$.

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + \langle f|g\rangle + \langle g|f\rangle$$

$$= \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\,(\langle f|g\rangle).$$

Since $\|f\| < \infty$, $\|g\| < \infty$ and $|\langle f|g\rangle| < \infty$ then

$$\|f+g\| < \infty \Rightarrow f+g \in L^2_w(a,b).$$

Remark: The example that we have studied can be used here: In this case

$\lim_{k,l \to \infty} \rho(|f_k\rangle, |f_l\rangle) = 0$ implies the existence of a function $g \in L^2_w(-1,1)$

so that $\lim_{k\to\infty} \rho(|g\rangle, |f_k\rangle) = 0$

$$\Rightarrow g = f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & -1 \leq x < 0 \end{cases}$$

(square integrable)

We have now the Riesz-Fisher theorem.

Theorem: $L^2_w(a,b)$ is a complete space

This means that any Cauchy sequence $\{|f_1\rangle, \dots |f_n\rangle, \dots\}$ in $L^2_w(a,b)$ converges to a function $|f\rangle$ in $L^2_w(a,b)$. Let $\{|f_k\rangle\}$, $k = 1, 2, \dots$ be a sequence satisfying

$\lim_{m,n\to\infty} \rho(|f_m\rangle, |f_n\rangle) = 0$ implies existence of $|f\rangle$

such that $\rho(|f\rangle, |f_k\rangle) \xrightarrow[k\to\infty]{} 0$ or if $\lim_{m,n\to\infty} \|f_m - f_n\| = 0 \Rightarrow \exists f \in L^2_w(a, w)$ such that

$$\lim_{m\to\infty} \|f - f_m\| = 0$$

f is unique, the integral we use is the Riemann integral this mean that, if there exist another function $g \in L^2_w(a,b)$ satisfying the same limit

$$\lim_{m\to\infty} \|g - f_n\| = 0$$

consider the triangular inequality

$$\|f - g\| \leq \|f - f_n\| + \|g - f_n\|$$

in the limit $RHS \to 0$ hence

$$\|f - g\| = 0 \quad \text{or} \quad \int_a^b w(x)|f - g|^2 dx = 0$$

For the riemann integral $|f - g| = 0$ on $[a, b]$

or $f = g$ over $[a, b]$. (Uniqueness)

4. Expansions in Orthogonal Polynomials

a) Simple cases: Assume that $f(x) \in L^2_w(a,b)$ has the form

$$|f\rangle = \sum_{i=1}^{N} f^i |e_i\rangle$$

where the N-vectors $|e_i\rangle$, $i=1,2,\dots,N$ satisfy

$$\langle e_i | e_j \rangle = \delta_{ij} \qquad (i,j = 1,2,\dots,N).$$

Then we have

$$f^k = \langle e_k | f \rangle \qquad , \; k = 1,2,\dots,N$$

or

$$f^k = \int_a^b \bar{e}_{(k)}(x)\, f(x)\, w(x)\, dx \quad , \quad k=1,2,\dots,N$$

where $e_k(x)$ functions representing the vectors $|e_k\rangle$.
This is an example resembling the case in $\mathbb{R}^N$.
We only need the evolution of the integrals on the RHS of the last equation.

This is a very special case. In the expansion (if possible)

$$|f\rangle = \sum_i f^i |e_i\rangle$$

we need infinite number of orthonormal vectors e_i, $i=1,2,\dots$

b) The general case: Let

$$|e_1\rangle, |e_2\rangle, \dots, |e_k\rangle, \dots \in L^2_\omega(a,b)$$

be an infinite sequence of orthonormal vectors

$$\langle e_i | e_j \rangle = \delta_{ij} \quad , \quad i,j = 1,2,\dots$$

Now define

$$f^i \equiv \langle e_i | f \rangle \equiv \int_a^b \bar{e}_i(x) f(x) \omega(x) dx$$

and then construct the vectors

$$|f_k\rangle = \sum_{i=1}^{k} \langle e_i | f \rangle |e_i\rangle \equiv \sum_{i=1}^{k} f^i |e_i\rangle$$

We have an infinite sequence of functions

$$|f_1\rangle, |f_2\rangle, \dots, |f_k\rangle, \dots$$

We have

1. $\langle f_k | f_k \rangle \equiv \|f_k\|^2 = \sum_{i=1}^{k} |f^i|^2$
2. $\langle f | f_k \rangle = \sum_{i=1}^{k} |f^i|^2$
3. $|\langle f | f_k \rangle|^2 \leq \langle f | f \rangle \langle f_k | f_k \rangle$

$$\Rightarrow \quad \sum_{i=1}^{k} |f^i|^2 \leq \langle f | f \rangle$$

in the limit also $\sum_{i=1}^{\infty} |f^i|^2 \leq \langle f | f \rangle < \infty$

$\Rightarrow$ all $|f_k\rangle$ belong to $L^2_\omega(a,b)$

The inequality

$$\sum_{i=1}^{\infty} |f^i|^2 \leq \|f\|^2$$

is called the Bessel inequality.

<u>Remark</u>: In the finite dimensional case

$$|\alpha\rangle = \sum_{i=1}^{N} a^i |e_i\rangle \qquad \text{for a vector } |a\rangle$$

one can obtain

$$\langle\alpha|\alpha\rangle = \sum_{i=1}^{N} |a^i|^2$$

It is easy to observe that

$$\sum_{i<N} |a^i|^2 \leq \langle\alpha|\alpha\rangle$$

This is the finite dimensional version of the Bessel inequality.

One has the equality sign when one take the components of $|\alpha\rangle$ wrt all the N orthonormal basis vectors. The inequality sign implies that some of these basis vectors are missing in the sum.

c) <u>Definition of the basis</u>: A sequence (finite or infinite) of orthonormal vectors.

$|e_1\rangle, \ldots, |e_k\rangle, \ldots$

is called a basis (or a complete set of vectors) of the space if the only vector orthogonal to all vectors of this sequence is <u>the trivial null vector</u>

$$\langle e_i | e_j \rangle = \delta_{ij} \quad , \quad i, j = 1, 2, \ldots$$

<u>Remark</u>: This condition is clearly satisfied by a set N orthonormal vectors in an N-dimensional space.

Given an orthonormal basis in a space with infinite number of dimensions, we are intending to write, as in the case of the finite dimensional case

$$|f\rangle \overset{?}{=} \sum^{\infty} f^i |e_i\rangle \qquad *$$

with

$$f^i \equiv \langle e_i | f \rangle$$

But this is, at present, a formal expression. We have not defined yet such infinite sums whether they are defined or not we don't know.

The convergence of an infinite sum means that the sequence of partial sums has a limit. In the above case this limit should be a vector belonging to the same space as the vectors that form the sequence of partial sums. Here the importance of the "completeness" arise. It assures that the expression (*) is a meaningfull expression.

We now prove a theorem that holds for any complete space. Here we shall concentrate ourselves to the space $L^2_w(a,b)$.

d) A Theorem on the Fourier Coefficients.

<u>Theorem</u>: Assume that there exist an orthonormal basis $|e_i\rangle$ $(i=1,2,\dots)$ in $L^2_w(a,b)$. Then, for any $|f\rangle \in L^2_w(a,b)$, the sequence of vectors

$$|f_k\rangle = \sum_{i=1}^{k} f^i |e_i\rangle$$

where $f^i \equiv \langle e_i|f\rangle = \int_a^b \bar{e}_{(i)}(x)\, f(x)\, w(x)\, dx$

has $|f\rangle$ as the limiting vector in the sense that

$$\lim_{k\to\infty} \rho(|f\rangle, |f_k\rangle) = 0$$

Proof: First let us check the CC.

$$\rho^2(|f_k\rangle, |f_l\rangle) = (\langle f_k| - \langle f_l|)(|f_k\rangle - |f_l\rangle)$$

$$= \sum_{l+1}^{k} \bar{f}^i \langle e_i| \sum_{l+1}^{k} f^j |e_j\rangle$$

$$= \sum_{l+1}^{k} |f^i|^2 = \sum_{i=1}^{k} |f^i|^2 - \sum_{i=1}^{l} |f^i|^2$$

here we assumed $k > l$

From the Bessel inequality

$$\sum_{i=1}^{\infty} |f^i|^2 < \infty$$

hence
$$\lim_{k\to\infty} \sum_{i=1}^{k} |f^i|^2 = \lim_{l\to\infty} \sum_{i=1}^{l} |f^i|^2$$

$$\Rightarrow \lim_{\substack{k\to\infty \\ l\to\infty}} \left(\sum_{i=1}^{k} |f^i|^2 - \sum_{i=1}^{l} |f^i|^2 \right)$$

$$= \lim_{k\to\infty} \sum_{i}^{k} |f^i|^2 - \lim_{l\to\infty} \sum_{i}^{l} |f^k|^2 = 0$$

Hence, since $L^2_w(a,b)$ is complete then there exist a vector $|g\rangle$ such that

$$\lim_{k\to\infty} \rho(|g\rangle, |f_k\rangle) = 0$$

The sequence converges to a function $|g\rangle$ which is also in $L^2_w(a,b)$. Let us prove that this vector is indeed $|f\rangle$.

$$| \langle g|e_i\rangle - \langle f_k|e_i\rangle | = |(\langle g| - \langle f_k|)|e_i\rangle|$$

$$\leq \rho(|f_k\rangle, |g\rangle) \langle e_i|e_i\rangle \to 0 \quad \text{as } k\to\infty$$

$$\Rightarrow \langle g|e_i\rangle = \lim_{k\to\infty} \langle f_k|e_i\rangle$$

$$= \langle f|e_i\rangle$$

$$(\langle g| - \langle f|)|e_i\rangle = 0 \Rightarrow |g\rangle = |f\rangle$$

since $|e_i\rangle$ form a basis the only vector orthogonal to the set $|e_i\rangle$ is the zero vec.

The numbers f^i are sometimes called the Fourier coefficients of $|f\rangle$ with respect to the basis

e) Parseval's relation: let $f \in L^2_w(a,b)$

An orthonormal set $|e_i\rangle$, $(i = 1, 2, \ldots)$ is a basis of $L^2_w(a,b)$ ~~then~~ iff

$$\langle f | f \rangle = \sum_{i=1}^{\infty} |f^i|^2$$

a) If $|e_i\rangle$ $(i = 1, 2, \ldots)$ is a basis of $L^2_w(a,b)$ then

$$\langle f | f \rangle = \sum_{i=1}^{\infty} |f^i|^2$$

Proof: let f_k, $k = 1, 2, \ldots$ be a sequence of partial sums

$$f_k = \sum^{k} f^i |e_i\rangle$$

since $L^2_w(a,b)$ is complete

$$\lim_{k \to \infty} \rho^2(|f\rangle, |f_k\rangle) = 0 \qquad \text{or}$$

$$\lim_{k \to \infty} (\langle f| - \langle f_k|)(|f\rangle - |f_k\rangle)$$

$$= \lim \Big[\langle f | f \rangle - \lim_{k \to \infty} (\langle f_k | f \rangle + \langle f | f_k \rangle) + \lim \langle f_k | f_k \rangle \Big]$$

$$= \lim \Big[\langle f | f \rangle - \sum_{i=1}^{k} |f^i|^2 \Big] = 0$$

$$\Rightarrow \quad \langle f|f\rangle = \sum_{i=1}^{\infty} |f^i|^2$$

If the ~~complete~~ orthonormal set of vectors $|e_i\rangle$, $i=1,2,\ldots$ in the Bessel's Inequality is a basis we have the equality and

Bessel's Inequality $\Rightarrow$ Parseval's relation.

b) If the Fourier coefficients satisfy the Parseval's relation then the set $|e_i\rangle$ is a basis

We have

$$\sum_{i=1}^{\infty} |f_i|^2 = \langle f|f\rangle$$

let $|e_i\rangle$ be not a basis only an orthonorm set. Hence there exist at least one vector $|a\rangle \neq 0$ should be added to the set $|e_i\rangle$ to make it a basis, Hence

$|a\rangle \neq 0$

$$\sum_{i=1}^{\infty} |f_i|^2 + |f_a|^2 \geq \langle f|f\rangle$$

But this contradiction to the Bessel's Inequality. Bessel's inequality for the orthonorm set

$|a\rangle, |e_1\rangle, \ldots, |e_n\rangle, \ldots$

we should have $\sum |f_i|^2 + |f_a|^2 \leq \langle f|f\rangle$

Hence $|e_i\rangle$ is the basis. Only 0 vector can be added.

we get $|f_a|^2 \geq 0$ and $|f_a|^2 \leq 0 \Rightarrow |f_a| = 0 \Rightarrow \langle a|f\rangle = 0$ $\forall f \Rightarrow |a\rangle = 0$ vector $\Rightarrow |e_i\rangle$ is a basis

Summary:

Given a set $e_{(i)}(x)$, $i=1,2,\ldots$ of orthonormal functions representing a set of basis vectors of $L^2_w(a,b)$, any function

$f(x) \in L^2_w(a,b)$.

can be expanded in the infinite limit

$$f(x) = \sum_{i=1}^{\infty} f^i e_{(i)}(x) \qquad (*)$$

with
$$f^i = \int_a^b \bar{e}_i(x) f(x) w(x) dx$$

The equality sign in $(*)$ means that the partial sums

$$f_k = \sum_{i=1}^{k} f^i e_{(i)}(x).$$

converge in the mean to $f(x)$.

$$\lim_{k\to\infty} \int_a^b \left| f(x) - \sum_{i=1}^{k} f^i e_{(i)}(x) \right|^2 w(x) dx = 0$$

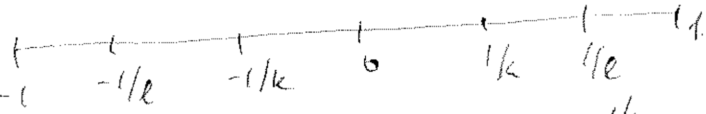
Appendix:

Proof of $\lim \rho^2(f_k, f_l) \to 0$ as $k, l \to \infty$ at page 27

$$\rho^2(f_k, f_l) = \int_{-1}^{1} |f_k - f_l|^2 dx$$

let $k > l$ (without loosing any generality).

Then



$$\int_{-1}^{1} |f_k - f_l|^2 dx = \int_{-1}^{-1/l} |0 - 0|^2 dx + \int_{-1/l}^{-1/k} \left| \frac{lx+1}{2} \right|^2 dx$$

$$+ \int_{-1/k}^{1/k} \left| \frac{kx+1}{2} - \frac{lx+1}{2} \right|^2 dx + \int_{1/k}^{1/l} \left| 1 - \frac{lx+1}{2} \right|^2 dx$$

$$+ \int_{1/l}^{1} |1 - 1|^2 dx$$

$$= \frac{1}{4} \int_{-1/l}^{-1/k} (l^2x^2 + 2lx + 1)\, dx + \frac{1}{4}(k-l)^2 \int_{-1/k}^{1/k} x^2 dx$$

$$+ \frac{1}{4} \int_{1/k}^{1/l} (l^2x^2 - 2lx + 1)\, dx$$

$$= \frac{1}{2}\left[-\frac{l^2}{3}\left(\frac{1}{k^3} - \frac{1}{l^3}\right) + l\left(\frac{1}{k^2} - \frac{1}{l^2}\right) - \frac{1}{k} + \frac{1}{l} \right]$$

$$+ \frac{1}{6}(k^2 - 2kl + l^2)\frac{1}{k^3}$$

$$= \frac{l}{6k^2} + \frac{1}{6l} - \frac{1}{3k}, \quad k > l.$$

$$\Rightarrow \lim_{k,l\to\infty} \rho^2(f_k, f_l) = \lim_{k,l\to 0} \left(\frac{l}{6k^2}\right), \quad k > l$$

$$l/k < 1$$

$$0 < \frac{l}{k^2} < \frac{1}{k}.$$

$$\Rightarrow \lim_{k,l\to\infty} (l/k^2) = 0.$$

$$\lim_{k,l\to\infty} \rho^2(f_k, f_l) = 0$$

Hence the sequence $\{f_k\}$ is Cauchy.